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BENDING OF CIRCULAR PLATES OF HARDENING MATERIAL

E. P. Popov, M. Khojasteh-Bakht and S. Yaghmai

University of California, Berkeley

Abstract—A general bending analysis of axisymmetrically loaded and supported circular plates of hardening material is presented in this paper. This is an extension of a previous work by the authors [13] on the analysis of elastic–plastic circular plates. The Kirchhoffean assumption is postulated and the investigation is limited to small deformations. For illustrative purposes, the incremental theory of plasticity with von Mises yield condition and the associated flow rule are adopted, and work hardening is assumed to be isotropic. This is not an inherent limitation of the method, as any plasticity hardening law can be used. The finite element approach using the direct stiffness method of matrix analysis of structures is employed to achieve the solution. The plate is divided into a number of annular elements which are further subdivided into several layers along their depths. Loads are applied in small finite increments. For each increment of loading, material properties are assigned to each layer of all annular elements and the stiffness matrix of the plate is computed accordingly. The variation of material properties within each increment of external load is considered. An example of a uniformly loaded clamped plate is given.

NOTATION

e_{ij}	deviatoric strain tensor
ε_{ij}	strain tensor
ε_{ij}^E	elastic components of strain tensor
ε_{ij}^P	plastic components of strain tensor
ε_i	principal strain components
E	modulus of elasticity
E_t	tangent modulus in uniaxial test
E_{ij}	elastic–plastic modulus
f	yield function or loading function
h	plate thickness
h_k	distance of kth layer from the reference plane
H	function of equivalent plastic strain
K_r, K_θ	radial and tangential curvatures
M_r, M_θ	radial and tangential moments per unit length
$2n$	number of layers
$p(r)$	transverse load per unit area
Q	radial transverse shear per unit length
r, θ, z	coordinate axes taken in the plate, z is measured positive downward from the reference plane
τ_{ij}	stress tensor
τ_i	principal stress components
ω	slope
w	deflection

INTRODUCTION

The problem of bending analysis of axisymmetrically loaded and supported circular plates made of hardening material has been studied in the past. Solutions based on both the deformation and the flow theory of plasticity have been reported by the investigators.

The early attempt was made by Sokolovsky [1] who used deformation theory of plasticity in the bending analysis of circular plates. The rigid-work hardening model together with piecewise linear yield conditions has been utilized in the solution of circular plates problems [2, 3]. The case of plates under uniform [2] and concentrated load [4] has been analyzed for material obeying Tresca yield condition. Mises yield condition has been linearized for the solution of plate problems [5, 6]. In addition to solid plates, the case of annular plate was treated in [7, 8]. The use of numerical integration to handle the bending of circular plates made of rigid-isotropic hardening material is demonstrated in [9, 10] and for rotational shell made of elastic-isotropic hardening in [11]. In the latter paper Marcel and Pilgrim used a step-by-step predictor–corrector approach and exhibited the propagation of the plastic region in a torispherical pressure vessel head. An incremental analysis of circular plates for Reuss–Mises stress–strain relationships was presented by Lackman [12]. In the paper an example for a uniformly loaded simply supported circular plate is given.

In this paper a perfectly general method of bending analysis of circular plates with axisymmetrical loading and support conditions is presented. The proposed procedure is applicable for any inelastic response but is limited to small deflections. In the proposed method of analysis any type of hardening material can be specified. For the purpose of illustration, however, incremental law of plasticity for isotropic hardening material obeying the von Mises yield condition has been selected. This paper is an extension of an earlier work [13] on circular plates of elastic perfectly plastic material.

FORMULATION OF THE PROBLEM

The general finite element approach using the stiffness method of matrix analysis of structures for circular plates has been presented previously [13]. This leads to the following equilibrium and strain displacement relations which are also applicable to the case under consideration here.

$$\frac{\mathrm{d}^2\Delta M_r}{\mathrm{d}r^2}+\frac{1}{r}\left(2\frac{\mathrm{d}\Delta M_r}{\mathrm{d}r}-\frac{\mathrm{d}\Delta M_\theta}{\mathrm{d}r}\right)+\Delta p(r) = 0 \tag{1}$$

$$\begin{Bmatrix}\Delta\varepsilon_r\\ \Delta\varepsilon_\theta\end{Bmatrix} = -z\begin{Bmatrix}\Delta K_r\\ \Delta K_\theta\end{Bmatrix};\qquad \begin{Bmatrix}\Delta K_r\\ \Delta K_\theta\end{Bmatrix} = \begin{Bmatrix}\dfrac{\mathrm{d}^2\Delta w}{\mathrm{d}r^2}\\ \dfrac{1}{r}\dfrac{\mathrm{d}\Delta w}{\mathrm{d}r}\end{Bmatrix} \tag{2}$$

where

ΔM_r, ΔM_θ respectively are the increments of radial and tangential moments per unit length, and $\Delta p(r)$ is the increment of applied load per unit area, see Fig. 1.

$\Delta\varepsilon_r$, $\Delta\varepsilon_\theta$ are radial and tangential strain increments.

ΔK_r, ΔK_θ are increments of curvature of middle plane, and Δw is increment of transverse deflection.

The plate is divided into a number of annular elements which are further subdivided into several layers of equal thickness symmetrically arranged with respect to middle plane, see Fig. 2. The material properties are assigned to each layer from the knowledge of load

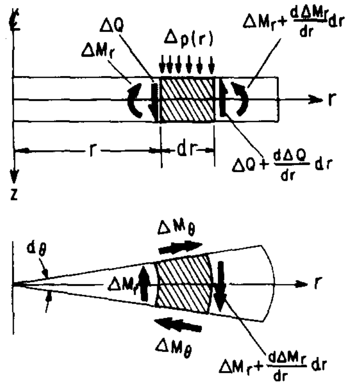

FIG. 1. Axisymmetric loading.

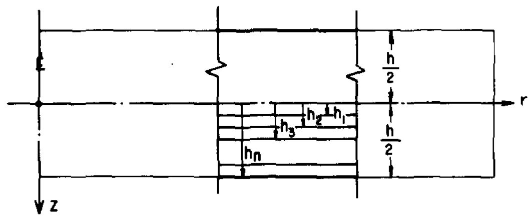

FIG. 2. Arrangement of layers along the depth of the plate.

history and on this basis the expression for $\{\Delta M\}$ can be formulated as follows:

$$\{\Delta M\} = 2 \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} [E^{(k)}]\{\Delta\varepsilon\}\, z\, \mathrm{d}z \tag{3}$$

or

$$\begin{Bmatrix} \Delta M_r \\ \Delta M_\theta \end{Bmatrix} = \{\Delta M\} = -[D]\{\Delta K\} \tag{3a}$$

where

$$\underset{2\times 2}{[D]} = \frac{h^3}{12n^3} \sum_{k=1}^{n} [E^{(k)}](3k^2 - 3k + 1) \tag{4}$$

$$[D] = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}; \qquad D_{12} = D_{21} \tag{5}$$

and

$E^{(k)}$ is the elastic–plastic modulus of kth layer, whose expression is developed in the next section.

Substitution of (3) into (1) leads to the following governing differential equation

$$\Delta w^{\mathrm{iv}} + \frac{2}{r}\Delta w^{\mathrm{iii}} - \left(\frac{\Lambda}{r}\right)^2 \left(\Delta w^{\mathrm{ii}} - \frac{1}{r}\Delta w^{\mathrm{i}}\right) = \frac{\Delta p(r)}{D_{11}} \tag{6}$$

where

$$\Lambda^2 = \frac{D_{22}}{D_{11}},$$

and superscripts denote differentiation with respect to r. The homogenous solution of (6) depends on parameter Λ.

(a) for $\Lambda = 1$

$$\Delta w = a_1 + a_2 r^2 + a_3 \ln r + a_4 r^2 \ln r \tag{7}$$

(b) for $\Lambda \neq 1$. $\Lambda \neq 0$

$$\Delta w = a_1 r^{1+\Lambda} + a_2 r^{1-\Lambda} + a_3 r^2 + a_4 \tag{8}$$

The case $\Lambda = 0$ is not encountered in this problem.

The homogenous solutions (7) and (8) are employed to establish the stiffness matrix of each element.

$$\underset{4\times 1}{\{\Delta S\}} = \underset{4\times 4}{[k]} \; \underset{4\times 1}{\{\Delta v\}} \tag{9}$$

where

$\{\Delta S\}$ and $\{\Delta v\}$ denote the increments of nodal ring forces and corresponding nodal ring displacements respectively, see Fig. 3.

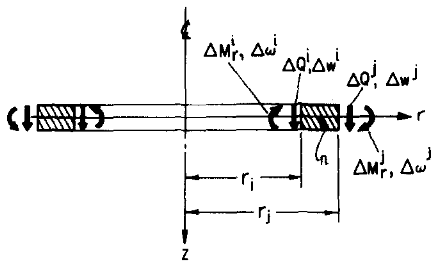

FIG. 3. Forces and displacements of a ring element.

$[k]$ is the element stiffness matrix which is given in the Appendix for the case when $\Lambda \neq 1$, $\Lambda \neq 0$.

Both tributary area approach and consistent equivalent nodal ring load method [14] have been employed to convert the transverse distributed load to nodal ring loads. Comparison of the results using these approaches indicates little difference. This is partly because of the small width of the elements used in the analysis for the purpose of assigning the material properties in each layer. The expressions for consistent equivalent nodal ring forces are given in [15].

The stiffness matrix of the whole plate is assembled by employing the direct stiffness method of matrix analysis of structures.

CONSTITUTIVE EQUATIONS

The main purpose of this paper is to apply the general method presented above to hardening materials whose constitutive law may be obtained either theoretically or experimentally. For the purpose of illustration a possible constitutive relation is presented in this section which can be used in the solution of plate problems.

The material is assumed to be time independent and initially free from residual stresses. The strain increments for small deformation can be written

$$\mathrm{d}\varepsilon_{ij} = \mathrm{d}\varepsilon_{ij}^{E} + \mathrm{d}\varepsilon_{ij}^{P} \tag{10}$$

where E and P respectively designate the elastic and plastic components of the strain tensor ε_{ij}.

The elastic component of strain increment is related to stress increment through generalized Hooke's Law which for isotropic materials can be written as

$$\mathrm{d}\varepsilon_{ij}^{E} = \frac{1+\nu}{E}\mathrm{d}\tau_{ij} - \frac{\nu}{E}\mathrm{d}\tau_{kk}\delta_{ij} \tag{11}$$

In order to establish the relations between the plastic components of strain and the state of stress, the existence of the plastic potential and the validity of the normality rule at a regular point on the yield surface are assumed. Thus

$$\mathrm{d}e_{ij}^{P} = \mathrm{d}\varepsilon_{ij}^{P} = \mathrm{d}\lambda\frac{\partial f}{\partial \tau_{ij}} \tag{12}$$

where e_{ij} is the deviatoric strain tensor, $\mathrm{d}\lambda$ is a non-negative function which may depend on stress, stress increment, strain and history of loading, and

$$f(\tau_{kk}, \varepsilon_{kl}^{P}, \kappa) = 0 \tag{13}$$

is the yield condition in which κ is a work hardening parameter.

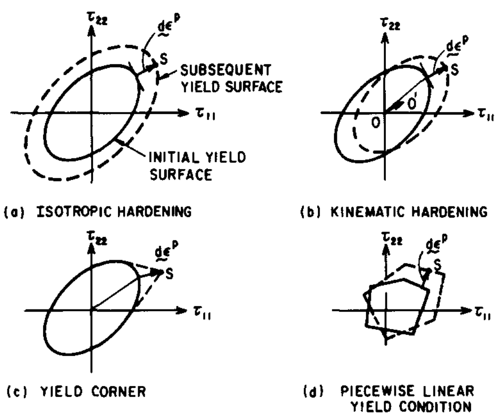

FIG. 4. Hardening rules.

Several hardening rules have been proposed to take into account the change in size and shape of the yield surface during plastic deformation. Isotropic hardening, see Fig. 4(a), which at progressively higher stresses exhibits uniform expansion of the initial yield surface, is the most widely used law to describe hardening. This rule, however, does not account for Bauschinger effect. To include this effect, the "kinematic hardening" rule was suggested by Prager [16], see Fig. 4(b), and was later modified by Ziegler [17]. Further suggestions such as piecewise linear yield conditions, which accommodate both translation and expansion of the yield surface, Fig. 4(d), and the concept of yield corner stating that the yield surface changes only locally, Fig. 4(c), have also been advanced Numerous tests have been conducted to check these theories. The results are contradictory and no definite conclusions have been reached so far. In this paper isotropic hardening rule was adopted although any other rule of plastic flow can be used.

To express the measure of hardening κ, two schemes are frequently used [18]. The first approach suggests that the degree of hardening is a function only of total plastic work

$$\kappa = \kappa(W_p), \qquad W_p = \int \tau_{ij}\,\mathrm{d}\varepsilon_{ij}^P \tag{14}$$

The second approach defines κ as

$$\kappa = \kappa\left(\int \mathrm{d}\bar{\varepsilon}^P\right), \qquad \mathrm{d}\bar{\varepsilon}^P = (\surd\tfrac{2}{3})(\mathrm{d}\varepsilon_{ij}^P\,\mathrm{d}\varepsilon_{ij}^P)^{\frac{1}{2}} \tag{15}$$

As pointed out by Hill [18], the above two concepts are equivalent for materials obeying von Mises yield condition.

Adopting the Mises yield condition and the stated assumptions, expression (13) can be stated as

$$f = \bar{\sigma} - H(\bar{\varepsilon}^P) = 0 \tag{16}$$

where

$$\bar{\sigma} = (\surd\tfrac{3}{2})(s_{ij}s_{ij})^{\frac{1}{2}} = \surd(3J_2)$$

and

$$\bar{\varepsilon}^P = \int_{\text{path}} \mathrm{d}\bar{\varepsilon}^P$$

in which

J_2 is the second invariant of deviatoric stress tensor

s_{ij} the deviatoric stress tensor

$\bar{\sigma}$ the effective stress, and

H a function of equivalent plastic strain

Combining equations (11), (15) and (16) we obtain the basic relation for the plastic strain increment

$$\mathrm{d}\varepsilon_{ij}^P = \frac{1}{H'}\frac{\partial\bar{\sigma}}{\partial\tau_{ij}}\frac{\partial\bar{\sigma}}{\partial\tau_{kl}}\mathrm{d}\tau_{kl}$$

or

$$\mathrm{d}\varepsilon_{ij}^P = \frac{3}{2}\frac{1}{H'}\frac{s_{ij}s_{kl}}{s_{mn}s_{mn}}\mathrm{d}\tau_{kl} \tag{17}$$

where $H' = \mathrm{d}\bar{\sigma}/\mathrm{d}\bar{\varepsilon}^P$ is the slope of the $\bar{\sigma}$–$\bar{\varepsilon}^P$ curve.

If the data for uniaxial tension test are used to define H', it can be shown that

$$\frac{1}{H'} = \frac{1}{E_t} - \frac{1}{E} \tag{18}$$

where E_t is the tangent modulus, and E the Young's modulus. If data from simple shear test are used, then

$$\frac{1}{H'} = \frac{1}{3}\left(\frac{1}{\mu_t} - \frac{1}{\mu}\right) \tag{19}$$

where μ_t is the tangent modulus in shear for the τ–γ diagram, and μ is the elastic shear modulus.

For H' to be invariant, the comparison of (18) and (19) leads to the following requirement

$$\frac{3}{E_t} - \frac{1}{\mu_t} = \frac{1-2\nu}{E} \tag{20}$$

Expression (20) imposes a restriction on E_t and μ_t, which generally does not hold true for all materials.

Substituting for $d\varepsilon_{ij}^E$ and $d\varepsilon_{ij}^P$ from (11) and (17) into (10), and considering the uniaxial tension test as the basis for determining H', we have

$$d\varepsilon_{ij} = S_{ijkl}\, d\tau_{kl} \tag{21}$$

where

$$S_{ijkl} = \frac{1+\nu}{2E}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{\nu}{E}\delta_{ij}\delta_{kl} + \frac{3}{2}\left(\frac{1}{E_t} - \frac{1}{E}\right)\frac{s_{ij}s_{kl}}{s_{mn}s_{mn}} \tag{22}$$

Inverting equation (21) and specializing for the case of plane stress, we obtain the following expressions

$$\begin{bmatrix} d\tau_1 \\ d\tau_2 \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \begin{bmatrix} d\varepsilon_1 \\ d\varepsilon_2 \end{bmatrix} \tag{23}$$

that are defined in the principal directions of stresses and strains and in which

$$E_{11} = E\frac{\zeta + (1-\zeta)(s_2/\bar{\sigma})^2}{(1-\nu^2)\zeta + (1-\zeta)[(s_1^2 + 2\nu s_1 s_2 + s_2^2)/\bar{\sigma}^2]}$$

$$E_{12} = E_{21} = E\frac{\nu\zeta - (1-\zeta)(s_1 s_2/\bar{\sigma}^2)}{(1-\nu^2)\zeta + (1-\zeta)[(s_1^2 + 2\nu s_1 s_2 + s_2^2)/\bar{\sigma}^2]} \tag{24}$$

$$E_{22} = E\frac{\zeta + (1-\zeta)(s_1/\bar{\sigma})^2}{(1-\nu^2)\zeta + (1-\zeta)[(s_1^2 + 2\nu s_1 s_2 + s_2^2)/\bar{\sigma}^2]}$$

where

$$\zeta = E_t/E; \qquad s_1 = \tau_{11} - \tfrac{1}{2}\tau_{22}; \qquad s_2 = \tau_{22} - \tfrac{1}{2}\tau_{11}$$

Equation (23) is applicable during loading, that is, when

$$\frac{\partial f}{\partial \tau_{ij}} \mathrm{d}\tau_{ij} > 0. \tag{25}$$

When unloading takes place,

$$\frac{\partial f}{\partial \tau_{ij}} \mathrm{d}\tau_{ij} < 0, \tag{26}$$

Hooke's law must be used:

$$\begin{bmatrix} \mathrm{d}\tau_1 \\ \mathrm{d}\tau_2 \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} \mathrm{d}\varepsilon_1 \\ \mathrm{d}\varepsilon_2 \end{bmatrix} \tag{27}$$

Expressions (25) and (26) are checked at each layer for each increment of load and the computer has the command to take the appropriate constitutive relations (24) or (27). In this manner reloading after any unloading is accommodated.

It should be noted that as E_t approaches zero, the components of E_{ij} given by (24) approach the values for an elastic perfectly plastic material described previously [13].

NUMERICAL EXAMPLE

The numerical analysis of the problems has been carried out by an IBM 7090–7094 digital computer using Fortran IV language.

The behavior of a clamped plate, Fig. 5, subjected to uniformly distributed load was analyzed. The assumed uniaxial stress–strain diagram of the plate material is shown in Fig. 6. Poisson's ratio was assumed to be $\nu = 0{\cdot}33$. The plate was divided into 18 elements which were arranged as indicated in Fig. 5. For the purpose of the analysis the thickness of the plate was divided into 40 layers. After inelasticity sets in, load increments of 15 and 10 psi were used.

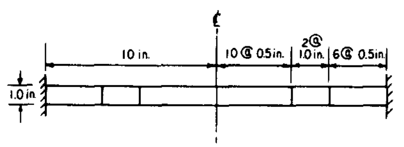

Fig. 5. Finite element idealization of the clamped plate.

The results of analysis are plotted in Figs. 7, 8, 9, and 10. The effect of inelastic behavior is evident from the presence of residual deflection after the removal of 560 psi load, see Fig. 7. Noticing the distribution of the residual moments from Figs. 8 and 9, the tendency for the redistribution of moments can be observed. In the example considered this is not of appreciable magnitude. The amount of residual moment depends on the hardening rate of the material. The pattern of redistribution of moments is similar to the results found previously for elastic–perfectly plastic material [13]. Since in the present method of analysis the variation and degree of inelasticity within each layer of an element is not distinguished, the calculated elastic–plastic boundaries consist of a series of small steps. These have been approximated by the curves shown in Fig. 10.

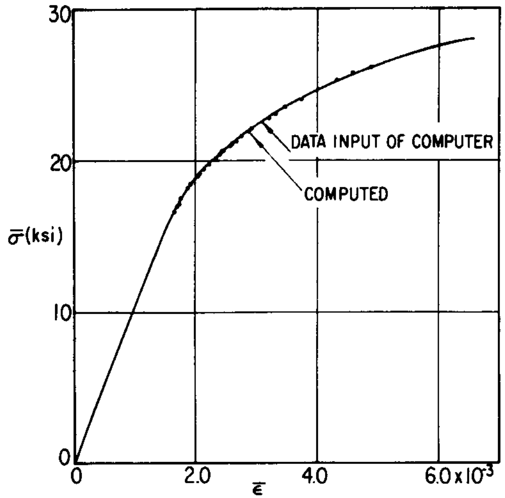

FIG. 6. Uniaxial stress–strain diagram of plate material.

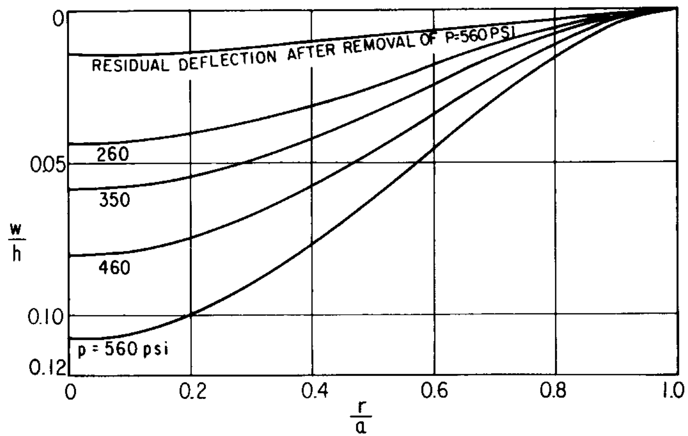

FIG. 7. Distributions of deflection.

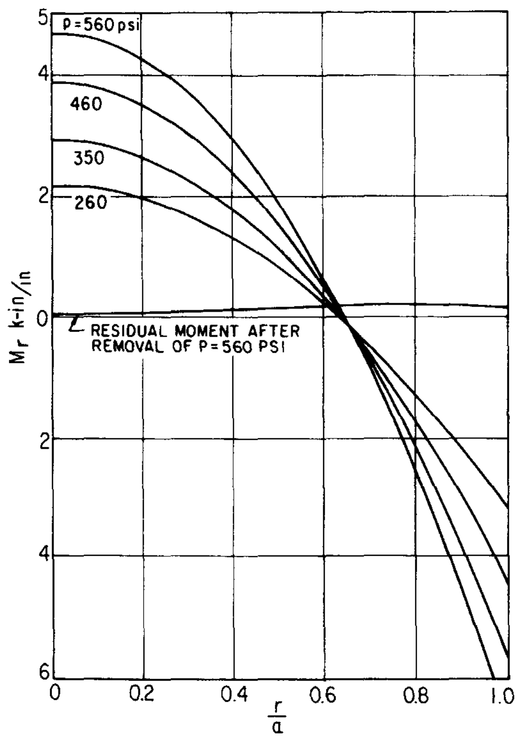

Fig. 8. Distributions of radial moment.

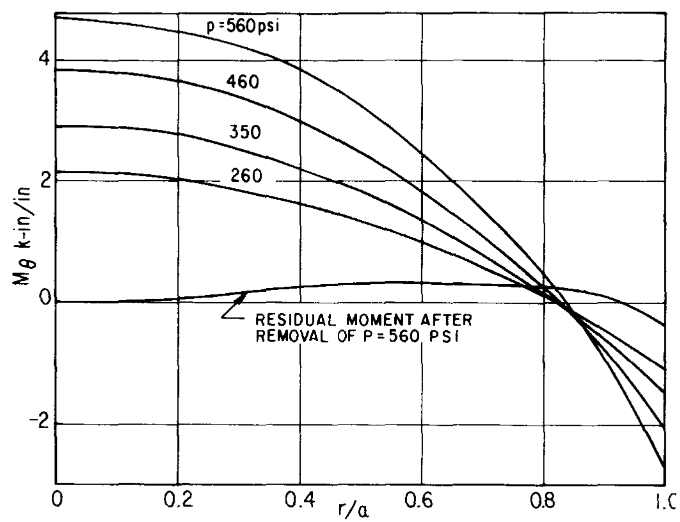

Fig. 9. Distributions of tangential moment.

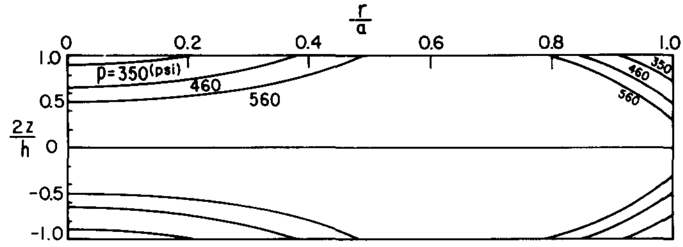

FIG. 10. Elastic–plastic boundaries.

The solid curve in Fig. 6 shows the original input uniaxial stress–strain diagram which has been used as the curve for the effective stress–strain relations. The small circles show the effective stress–strain relationship which is obtained from the stresses and strains in the plate. The coincidence of these two is a good indication of the rate of convergence of the solution.

A numerical difficulty was encountered at the initiation of plastic deformation in an element, that is when the value of Λ is close to unity. This causes an ill-conditioning of $[k]$ matrix. To overcome this difficulty two procedures were employed. The first made use of a series expansion in terms of $(1-\Lambda)$ and the factor $(1-\Lambda)/(1-\Lambda)$ causing ill-conditioning was removed. The second made use of a double precision field which retains 16 decimal digits. It was found that the second approach was superior to the first one, partly because of series truncation and the more involved formulas used in the first approach.

The execution time of the program depends mostly on the number of elements in the plate and the number of load increments. The number of layers in the elements does not affect the time consumption appreciably. For a plate with 16 elements and 40 layers about 12 sec are used for the execution of each load increment.

CONCLUSIONS

A general method of elastic plastic bending analysis of axisymmetrically loaded and supported circular plates has been presented in this paper. The method has a great potential for applications to axisymmetrical shell problems. The extension of this approach to shells of revolution is in progress.

In addition to the example presented in this paper, additional cases have been analyzed [15]. It was found that the sensitivity to the magnitude of load increments increases as the rate of hardening decreases, but that in general the results are not unduly sensitive to the magnitude of load increments. In this sense the results indicate a high degree of convergence of the solution.

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APPENDIX

Elements of matrix $[k]$ for a ring element when $\Lambda \neq 1, \Lambda \neq 0$ are as follows:

$$\underset{4\times 1}{\{\Delta S\}} = \underset{4\times 4}{[k]} \ \underset{4\times 1}{\{\Delta v\}}$$

$$k_{11} = \frac{2D_{11}}{r_i^2 r_j \beta}(1-\Lambda^2)^2\left[\left(\frac{r_j}{r_i}\right)^{\Lambda} - \left(\frac{r_i}{r_j}\right)^{\Lambda}\right]$$

$$k_{12} = \frac{2D_{11}}{r_i^2 \beta}(1-\Lambda^2)\left[(1-\Lambda)\left(\frac{r_i}{r_j}\right)^{1+\Lambda} - (1+\Lambda)\left(\frac{r_i}{r_j}\right)^{1-\Lambda} + 2\Lambda\right]$$

$$k_{13} = -k_{11}$$

$$k_{14} = \frac{2D_{11}}{r_i r_j \beta}(1-\Lambda^2)\left[(1-\Lambda)\left(\frac{r_j}{r_i}\right)^{1+\Lambda} - (1+\Lambda)\left(\frac{r_j}{r_i}\right)^{1-\Lambda} + 2\Lambda\right]$$

$$k_{21} = k_{12}$$

$$k_{22} = \frac{D_{11}}{r_i \beta}(1-\Lambda^2)\left\{(1-\Lambda)\left[\left(\frac{r_j}{r_i}\right)^{1+\Lambda} - \left(\frac{r_i}{r_j}\right)^{1+\Lambda}\right] - (1+\Lambda)\left[\left(\frac{r_j}{r_i}\right)^{1-\Lambda} - \left(\frac{r_i}{r_j}\right)^{1-\Lambda}\right]\right\} - \frac{D_{11}+D_{12}}{r_i}$$

$$k_{23} = -k_{21}$$

$$k_{24} = \frac{D_{11}}{r_j \beta}(1-\Lambda^2)\left\{(1-\Lambda)\left[\left(\frac{r_j}{r_i}\right)^{1-\Lambda} - \left(\frac{r_j}{r_i}\right)^{1+\Lambda} - \left(\frac{r_j}{r_i}\right)^{2} + 1\right] + (1+\Lambda)\left[\left(\frac{r_j}{r_i}\right)^{1-\Lambda} - \left(\frac{r_j}{r_i}\right)^{1+\Lambda} + \left(\frac{r_j}{r_i}\right)^{2} - 1\right]\right\}$$

$$k_{31} = \left(\frac{r_i}{r_j}\right)k_{13}$$

$$k_{32} = \left(\frac{r_i}{r_j}\right)k_{23}$$

$$k_{33} = -k_{31}$$

$$k_{34} = -\frac{r_i}{r_j}k_{14}$$

$$k_{41} = -k_{34}$$

$$k_{42} = \left(\frac{r_i}{r_j}\right)k_{24}$$

$$k_{43} = k_{34}$$

$$k_{44} = \frac{D_{11}+D_{12}}{r_j}+\frac{D_{11}}{r_j\beta}(1-\Lambda)^2\left\{(1-\Lambda)\left[\left(\frac{r_j}{r_i}\right)^{1+\Lambda}-\left(\frac{r_i}{r_j}\right)^{1+\Lambda}\right]-(1+\Lambda)\left[\left(\frac{r_j}{r_i}\right)^{1-\Lambda}-\left(\frac{r_i}{r_j}\right)^{1-\Lambda}\right]\right\}$$

where

$$\beta = (1-\Lambda)^2\left[\left(\frac{r_j}{r_i}\right)^{1+\Lambda}-\left(\frac{r_i}{r_j}\right)^{1+\Lambda}\right]^2-(1+\Lambda)\left[\left(\frac{r_j}{r_i}\right)^{1-\Lambda}-\left(\frac{r_i}{r_j}\right)^{1-\Lambda}\right]^2.$$

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Résumé—Une analyse générale de la flexion de plaques circulaires de matériau en cours de durcissement à charge et support symétriques à l'axe est présentée dans cet exposé. Ceci est un prolongement d'une oeuvre précédente des auteurs [13] sur l'analyse de plaques circulaires élastiques plastiques. La supposition de Kirchhoff est postulée et l'enquête se limite aux petites déformations. Dans un but d'illustration, la théorie d'accroissement de plasticité avec la condition de von Mises à la limite d'élasticité et la règle d'écoulement associée sone adoptées, et l'écrouissage est supposé être isotrope. Ceci ne présente pas une limite à la méthode, étant donné que n'importe quelle loi sur le durcissement plastique peut être utilisée. Le rapprochement par l'élément limité en utilisant la méthode directe de rigidité par l'analyse des matrices des structures est employée pour obtenir la solution. La plaque est divisée en un certain nombre d'éléments annulaires qui sont de plus divisés en plusieurs couches le long de leurs profondeurs. Des charges sont appliquées par petits accroissements limités. Pour chaque accroissement de charge, des qualités de matériau sont assignées à chaque couche de tous les éléments annulaires et la matrice de rigidité de la plaque est calculée en conséquence. La variation des qualités de matériau au cours de chaque accroissement de la charge extérieure est considérée. Un exemple de plaque serrée à charge uniforme est donné.

Zusammenfassung—Eine allgemeine Biegungsanalyse axialsymmetrisch belasteter und gestützter runder Platten aus gehärtetem Material wird in dieser Arbeit gegeben. Dies ist eine Erweiterung einer früheren Arbeit de Authoren [13] über die Analyse elastoplastischer runder Platten. Die Kirchhoff'sche Voraussetzung wird angenommen und die Untersuchung ist auf kleine Verformungen beschränkt. Zur Illustration werden die Zuwachstheorie der Plastizität mit der von Mises'schen Bedingung und die damit verbunderne Fliessregel angewandt und die Verfestigung wird als anisotropisch vorausgesetzt. Dies ist keine Eigenbeschränkung der Methode da jedes Härtungsgesetz angewandt werden kann. Zur Lösung wird das endliche Element mit der direkten Steifigkeitsmethode der strukturellen Matritzenanalyse angewandt. Die Platte wird in Ringe aufgeteilt, die weiter in verschiedene Lagen, der Tiefe nach, geteilt werden. Die Belastung wird in kleinen endlichen Zunahmen aufgebracht. Für jede Belastungszunahme werden für jede Lage jedes Teiles Materialeigenschaften bestimmt, und die Steifigkeitsmatritze wird entsprechend errechnet. Die Verschiedenheiten der Materialeigenschaften für die gegebenen Belastungszunahmen werden untersucht. Als Beispiel wird eine gleichmässig belastete gespannte Platte gegeben.

Абстракт—В работе представлен общий расчет изгиба осесиметрично нагруженной и опертой круглой пластинки из материала с упрочнением. Работа является продолжением предыдущей работы авторов [13], касающейся расчета упруго-пластичных круглых пластинок. Принимается допущение Кирхгоффа. Исследования ограничены малыми деформациями. Для иллюстрации принята теория пластического течения с условием текучести Мизеса и ассоциированным законом течения. Упрочнение предлагается изотропным. Это не является существенным ограничением метода, так как может быть использован любой закон пластического упрочнения. Задача решена методом конечных элементов при использовании прямого метода матричного исчисления, известного из строительной механики, для представления коэффициентов жесткости. Пластинка разделена на некоторые число кольцевых элементов, котоьые далее подразделены на несколько слоев по толщине. Прилагаемая нагрузка увеличивается малыми конечными приращениями. При каждом приращении нагрузки опеделены свойства материала каждого слоя во всех кольцевых элементах и соответственно расчитана матрица коэффициентов жесткости пластинки. Исследуется изменение свойств материала для каждого приращения внешней нагрузки. Приводится пример расчета равномерно нагруженной, заделаной пластинки.